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A special solution of the Laplace equation and the vase functions for the description of the planet's gravitational field

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A set of linear harmonic functions was obtained as a special solution of the Laplace equation and was used as a basis of the non-orthogonal functions for approximation of the gravitational potential. Maxwell method was applied to the construction of such a set of the linear harmonic functions. Resulting formulas were obtained in the form of the finite sums that provides the practical constructions of a new set of linear harmonic functions for global and regional gravity field description.

СПЕЦИАЛЬНОЕ РЕШЕНИЕ УРАВНЕНИЯ ЛАПЛАСА И БАЗИСНЫЕ ФУНКЦИИ ДЛЯ АППРОКСИМАЦИИ ГРАВИТАЦИОННОГО ПОТЕНЦИАЛА ПЛАНЕТ, Фоца Р., Абрикосов О. – в качестве неортогональных базисных функций для аппроксимации гравитационного потенциала планет используется семейство линейных гармонических функций, полученных на основании специального решения уравнения Лапласа. Для их построения использован метод Максвелла. Полученные таким образом формулы, в виде конечных сумм, позволяют практически реализовать построение нового семейства линейных гармонических функций для использования при аппроксимации гравитационного потенциала планет в глобальном и региональном масштабах.

СПЕЦІАЛЬНИЙ РОЗВ'ЯЗОК РІВНЯННЯ ЛАПЛАСА ТА БАЗИСНІ ФУНКЦІЇ ДЛЯ АПРОКСИМАЦІЇ ГРАВІТАЦІЙНОГО ПОТЕНЦІАЛУ ПЛАНЕТ, Фоца Р., Абрикосов О. – В якості неортогональних базисних функцій для апроксимації гравітаційного потенціалу планет використано сімейство лінійних гармонічних функцій, отриманих на основі спеціального розв'язку рівняння Лапласа. Для його побудови використано метод Максвела. Отримані таким чином формули, у вигляді скінчених сум, дозволяють практично реалізувати побудову нового сімейства лінійних гармонічних функцій для використання при апроксимації гравітаційного потенціалу планет в глобальному і регіональному масштабах.

For the global approximation of the gravity potential of the Earth, the Moon and other planets the series of solid spherical harmonics are used traditionally. Additionally today the non-orthogonal base functions, especially potentials of non-central radial multipoles are used also for the regional gravity field approximation. Below we shall consider some aspects of the gravity field approximation by a set of non-orthogonal base functions.

In (Hobson, 1931) the special solution of the Laplace equation was derived. In the central case

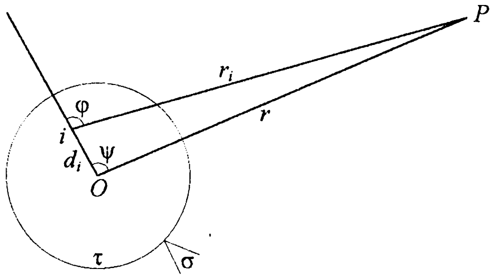

Fig. 1 To the explanation of the formula (1)

this solution may be written as

$$\widetilde{F} = \frac{1}{r_i}\ln\frac{r_i + r_i\cos\varphi}{2r_i^2}, \tag{1}$$

where r_i is the distance from fixed point i to current point P; φ is the angle between the directions Oi and iP (see Figure 1);

$$\cos\varphi = \frac{r\cos\psi - d_i}{r_i}; \tag{2}$$

and r_i is the distance from fixed point O to the current point P; d_i is the distance from point O to point i; ψ is the angle between the directions OP and Oi. The function $\widetilde{F}$ is the harmonic function in the outer space σ.

For construction of the set of base functions which based on the special solution (1) of Laplace equation we shall apply the method of Maxwell. This method is based on the fact that if a solution of the Laplace equation is known then any outer solutions may be derived by means of appropriate differentiating of it with respect to rectangular coordinates. In our case such differentiation may be replaced by the differentiation with respect to the direction d_i.

Let $\widetilde{Q}_n$ is the polynomial, which was obtained by n-times differentiating of the expression (1) with respect to the direction d_i

$$\widetilde{Q}_n = \frac{1}{n!}\frac{\partial^n}{\partial d_i^n}\left[\frac{1}{r_i}\ln\frac{r_i + r_i\cos\varphi}{2r_i^2}\right].$$

According to the Leibnitz formula we can write he expression (3) in the form

$$\widetilde{Q}_n = \frac{1}{n!}\sum_{m=1}^{n} C_n^m \frac{\partial^m}{\partial d^m}\left(\frac{1}{r_i}\ln\frac{r_i + r_i\cos\varphi}{2r_i^2}\right)\frac{\partial^{n-m}}{\partial d^{n-m}}\left(\frac{1}{r_i}\right), \tag{4}$$

where C_n^m are the binomial coefficients

$$C_{n-1}^m = \frac{(n-1)!}{m!(n-m-1)!}. \tag{5}$$

Besides, let

$$V_0 = 1/r_i \tag{6}$$

is the potential of non-central radial multipole of zero degree and let V_n is the potential of the multipole of degree n:

$$nV_n = (2n-1)\frac{\cos\varphi}{r_i} - (n-1)V_{n-2}\left(\frac{1}{r_i}\right)^2, \; V_n \equiv 0 \text{ for } n < 0. \tag{7}$$

Then

$$\frac{\partial^n}{\partial d_i^n}\left(\frac{1}{r_i}\right)=\frac{\partial^n}{\partial d_i^n}(V_0)=n!V_n \tag{8}$$

On the basis of these expressions we can derive

$$\frac{\partial}{\partial d_i}\left(\ln\frac{r_i+r_i\cos\varphi}{2r_i^2}\right)=\frac{2\cos\varphi-1}{r_i}=2\frac{1}{r_i}\cos\varphi-\frac{1}{r_i}, \tag{9}$$

$$\frac{\partial^2}{\partial d_i^2}\left(\ln\frac{r_i+r_i\cos\varphi}{2r_i^2}\right)=\frac{\partial}{\partial d_i}\left[2\frac{1}{r_i}\cos\varphi-\frac{1}{r_i}\right]=\frac{\partial}{\partial d_i}\left(2\frac{1}{r_i}\cos\varphi\right)-\frac{\partial}{\partial d_i}\frac{1}{r_i}, \tag{10}$$

$$\frac{\partial^n}{\partial d_i^n}\left(\ln\frac{r_i+r_i\cos\varphi}{2r_i^2}\right)=\frac{\partial^{n-1}}{\partial d_i^{n-1}}\left(2\frac{1}{r_i}\cos\varphi\right)-\frac{\partial^{n-1}}{\partial d_i^{n-1}}\frac{1}{r_i}. \tag{11}$$

Application of Leibnitz formula to the expression (11) yields

$$\frac{\partial^n}{\partial d_i^n}\left(\ln\frac{r_i+r_i\cos\varphi}{2r_i^2}\right)=-\frac{\partial^{n-1}}{\partial d_i^{n-1}}\left(\frac{1}{r_i}\right)+2\sum_{m=0}^{n-1}C_{n-1}^m\frac{\partial^{n-m-1}}{\partial d^{n-m-1}}\left(\frac{1}{r_i}\right)\frac{\partial^m}{\partial d_i^m}\cos\varphi. \tag{12}$$

The derivative of order m of $\cos\varphi$ with respect to d_i may be expressed as

$$\frac{\partial^m}{\partial d_i^m}\cos\varphi=m!\left[V_m r_i\cos\varphi-V_{m-1}\right]. \tag{13}$$

Let us denote

$$L_n=\frac{\partial^n}{\partial d_i^n}\left(\ln\frac{r_i+r_i\cos\varphi}{2r_i^2}\right). \tag{14}$$

Taking into account the expressions (8), (13) and (5) we can write the expressions for L_n:

$$L_0=\ln\frac{r_i+r_i\cos\varphi}{2r_i^2}, \tag{15}$$

$$L_n=-(n-1)!\left[V_{n-1}+2\sum_{m=0}^{n-1}V_{n-m-1}\left(V_{m-1}-V_m r_i\cos\varphi\right)\right]\qquad(n>0). \tag{16}$$

On the basis of the formulas (4), (8) and (14) we can write the final expression for $\widetilde{Q}_n$ of arbitrary degree n:

$$\widetilde{Q}_n=\sum_{m=0}^{n}\frac{1}{m!}V_{n-m}L_m. \tag{17}$$

Here L_m may be computed from (15), (16) and V_{n-m} – from (6), (7).

Obtained formulas leads to a theoretical construction of new set of harmonic functions (3) for further approximation of the gravity potential. The normalized $\widetilde{Q}_n$ for $n=0,1,\ldots 5$ and $d_i=0.7$ are illustrated by Figure 2.

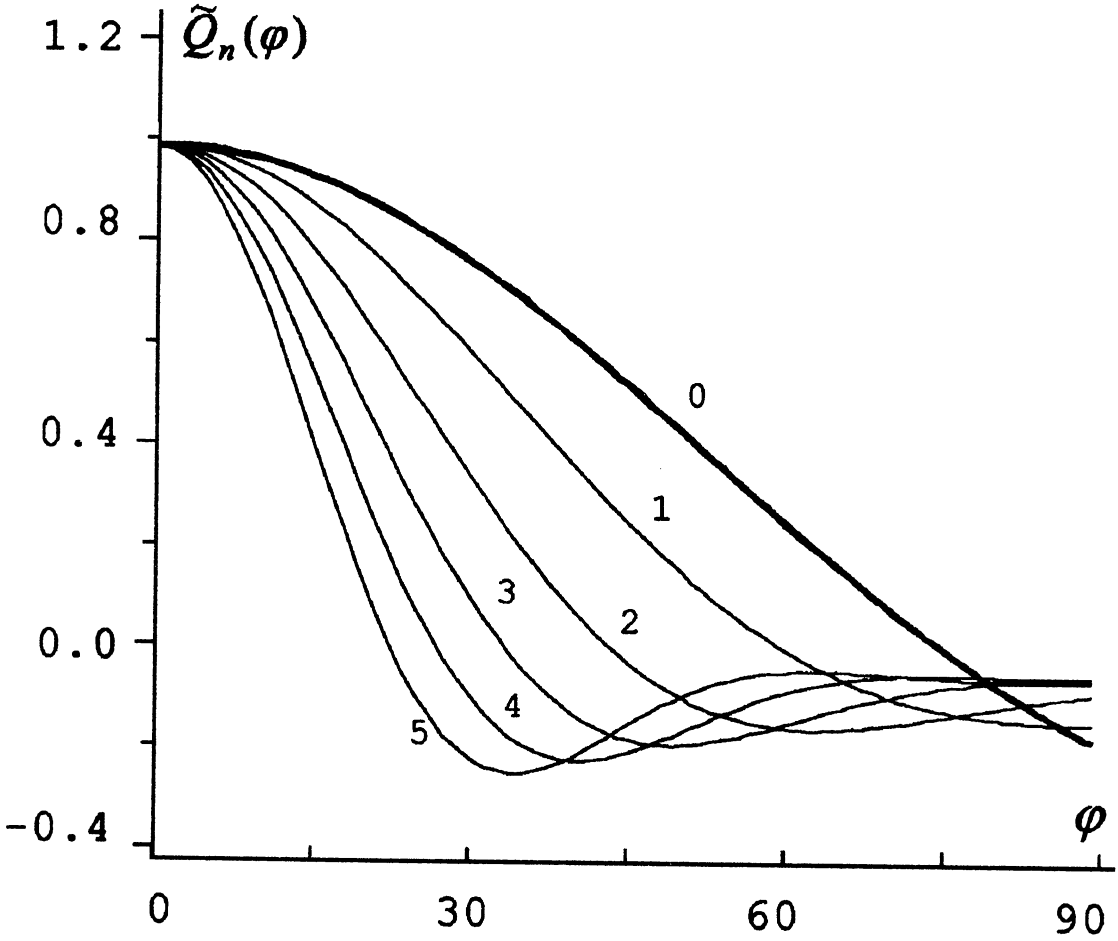

Figure 2. Normalized functions $\widetilde{Q}_n(\varphi)$ for $n = 1, 2, 3, 4, 5$ and $d_i = 0.7$

1. Hobson, E.W. (1931). *The theory of Spherical and Ellipsoidal Harmonics*, Cambridge.

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